1. Consider the convolutional code generated by \( g_0 = (101) \), \( g_1 = (111) \) and \( g_2 = (111) \).

(a) Construct a state and a trellis diagram for this code.

(b) Find the transfer-function of the code.

(c) Derive and plot upper-bounds to the performance of a ML soft-decision decoder and that of a ML hard-decisions decoder.

(d) Find the estimated message sequence \( \hat{x} \) when the received soft-data vector (each bit is transmitted using BPSK, with a 0 mapped to -1). Assume the initial encoder state is the all zero state.

\[
r = \begin{bmatrix} (-0.9, -0.2, -0.5) & (-0.5, 0.2, -0.3) & (0.1, -0.2, -0.1) & (-0.8, -0.2, 0.3) & (0.2, -0.2, -0.5) & (-0.7, -0.2, -0.1) \end{bmatrix}
\]

is observed. What would the decision of a hard-decisions decoder be? Show your computations on an appropriate trellis.

Solution to Problem 1

(a) The trellis and state diagrams are given below:

![Trellis Diagrams](image)

Figure 1: The state and trellis diagrams.

(b) After some manipulation, the transfer function is:

\[
T(D) = \frac{D^8 (1 - D^2 + D)}{1 - 2D^2 + D^4 - D^3} = D^8 + D^9 + 3D^{11} + 2D^{12} + 6D^{13} + 6D^{14} + 11D^{15} + 16D^{16} + 22D^{17} + 37D^{18} + 49D^{19} \ldots
\]
(c) The transfer-function bounds are given by:

\[ P(e) \leq T(D) \big|_{D = e^{-E/N_0}} \]  
soft – decisions

\[ P(e) \leq T(D) \big|_{D = \sqrt{4p(1-p)}} \]  
hard – decisions

and they are plotted below:

(d) Figure 4 shows the decoder trellis for both soft and hard decision decoding, as well as the decoded sequences in both cases.

2. Consider a fading channel in which the received data is

\[ r(t) = \alpha \sum_{k=0}^{N-1} a_k p(t - kT) + n(t), \]
where $\alpha$ is a zero-mean, Gaussian random variable with variance $\sigma^2$, $\mathbf{a}$ is a vector of modulation symbols with $a_k \in \{1, -1\}$, $n(t)$ is a zero-mean, white Gaussian process with spectral density $N_0/2$, and $p(t)$ has energy $E$ and $\mathcal{F}^{-1}\{|P(f)|^2\}$ is a Nyquist pulse.

(a) Design an optimum receiver that decides what the transmitted sequence $\mathbf{a}$ is.

(b) Now, consider the special case when $N = 2$ and a rate $1/2$ coded system that assigns information bits to transmitted symbols according to:

\[
\begin{align*}
0 & \iff 1 \ 1 \\
1 & \iff 1 \ -1
\end{align*}
\]

Derive an expression for the probability of error, and plot it for SNR’s from 0 to 20dB.

**Solution to Problem 2**

(a) The likelihood function can be obtained by taking an expectation of the conditional likelihood function
(given the fading), with respect to the fading:

\[ L(a) = E_r \left[ e^{\frac{2\alpha}{N_0} \sum_{k=0}^{N-1} a_k r_k - \frac{2\alpha^2}{N_0} \sigma^2} \right], \]

where

\[ r_k = \int_{-\infty}^{\infty} r(t) p(t - kT) dt \]

and we have made use of the Nyquist signaling property. Performing the expectation, taking logarithms, and dropping unnecessary terms we obtain

\[ \ell(a) = \left( \sum_{k=0}^{N-1} a_k r_k \right)^2. \]

(b) In this case, the optimum detector becomes:

\[(r_0 + r_1)^2 \geq (r_0 - r_1)^2\]

or, equivalently,

\[ r_0 \cdot r_1 \geq 0. \]

The probability of error can be argued to be the same whether a 0 or a 1 is sent. Assuming that a 0 is sent, the probability of error is:

\[ P(e) = \Pr(r_0 \cdot r_1 \leq 0) = \Pr(x \cdot y \leq 0), \]

where we define \( x = r_0/E \) and \( y = r_1/E \). It is easy to show that \( x \) and \( y \) are jointly Gaussian, having zero mean, variances \( \sigma^2 + \frac{N_0}{2E} \), and \( E [x \cdot y] = \sigma^2 \). Then,

\[ P(e) = \int_{\{x>0,y<0\} \cup \{x<0,y>0\}} f_{X,Y}(x,y)dx dy = \frac{1}{2} - \frac{1}{\pi} \sin^{-1} \left( \frac{2SNR}{2SNR + 1} \right), \]

where the last equality above can be derived through integration (see also the stochastic processes book by Papoulis, for example.)

(c) The ML estimate of \( \alpha \) based on the data \( r_0 \) in the first bit is obtained by maximizing the log-likelihood function:

\[ \ell(\alpha) = \alpha \int_{-\infty}^{\infty} r(t) p(t) dt - \frac{1}{2} \alpha^2 E. \]

Taking the derivative w.r.t \( \alpha \) and setting to zero yields

\[ \hat{\alpha} = \frac{r_0}{E}. \]

Assuming the estimate is perfect, the detector that decides whether the second bit is a 0 or 1 is

\[ \hat{\alpha} r_1 - \frac{1}{2} \hat{\alpha}^2 E \geq 0 - \hat{\alpha} r_1 - \frac{1}{2} \hat{\alpha}^2 E \]

from which

\[ \hat{\alpha} r_1 \geq 0. \]

Substituting \( r_0/E \) for \( \hat{\alpha} \), yields exactly the same receiver as the optimum, in this case. Its performance is the same as for the optimum.
Figure 5: Error probability vs average SNR.